Coherent states of the harmonic oscillator

In these notes I will assume knowledge about the operator method for the harmonic oscillator corresponding to sect. 2.3 i "Modern Quantum Mechanics" by J.J. Sakurai. At a couple of places I refefer to this book, and I also use the same notation, notably x and p are operators, while the correspondig eigenkets are $|x'\rangle$ etc.

1. What is a coherent state ?

Remember that the ground state $|0\rangle$, being a gaussian, is a minimum uncertainty wavepacket:
Proof:

$$\begin{aligned} x^2 &= \frac{\hbar}{2m\omega}(a + a^\dagger)^2 \\ p^2 &= -\frac{m\omega\hbar}{2}(a - a^\dagger)^2 \end{aligned}$$

Since

$$\begin{aligned} \langle 0|(a + a^\dagger)(a + a^\dagger)|0\rangle &= \langle 0|aa^\dagger|0\rangle = 1 && (1) \\ \langle 0|(a - a^\dagger)(a - a^\dagger)|0\rangle &= -\langle 0|aa^\dagger|0\rangle = -1 && (2) \end{aligned}$$

it follows that

$$\langle x^2\rangle_0 \langle p^2\rangle_0 = -\frac{\hbar^2}{4}1(-1) = \frac{\hbar^2}{4}$$

and finally since $\langle x\rangle_0 = \langle p\rangle_0 = 0$, it follows that

$$\langle(\Delta x)^2\rangle_0 \langle(\Delta p)^2\rangle_0 = \frac{\hbar^2}{4} \qquad (3)$$

We can now ask whether $|n\rangle$ is also a minimum uncertainty wave packet. Corresponding to (1) and (2) we have

$$\langle n|(a + a^\dagger)(a + a^\dagger)|n\rangle = \langle n|aa^\dagger + a^\dagger a|n\rangle = \langle n|2a^\dagger a + [a, a^\dagger]|n\rangle = 2n + 1$$

and similarly

$$\langle n|(a - a^\dagger)(a - a^\dagger)|n\rangle \;\; = -(2n + 1)$$

which implies

$$\langle(\Delta x)^2\rangle_n \langle(\Delta p)^2\rangle_n = \frac{\hbar^2}{4}(2n + 1)^2 \qquad (4)$$

so $|n\rangle$ is not minimal !

Clearly a crucial part in $|0\rangle$ being a minimal wave packet was

$$a|0\rangle = 0 \quad \Rightarrow \quad \langle 0|a^\dagger a|0\rangle = 0$$

this corresponds to a sharp eigenvalue for the *non-Hermitian* operator $m\omega x + ip$ even though, as we saw, there was (minimal) dispersion in both x and p. It is natural to expect other minimal wave-packets with non zero expectation values for x and p but still eigenfunctons of a, *i.e.*

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{5}$$

which implies $\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$. It is trivial to check that this indeed defines a minimal wave-packet

$$\begin{aligned}
\langle\alpha|(a + a^\dagger)|\alpha\rangle &= (\alpha + \alpha^\star) \\
\langle\alpha|(a - a^\dagger)|\alpha\rangle &= (\alpha - \alpha^\star) \\
\langle\alpha|(a + a^\dagger)(a + a^\dagger)|\alpha\rangle &= (\alpha + \alpha^\star)^2 + 1 \\
\langle\alpha|(a - a^\dagger)(a - a^\dagger)|\alpha\rangle &= (\alpha - \alpha^\star)^2 - 1
\end{aligned}$$

from which follows

$$\begin{aligned}
\langle(\Delta x)^2\rangle_\alpha = \langle x^2\rangle_\alpha - \langle x\rangle_\alpha^2 &= \frac{\hbar}{2m\omega} \\
\langle(\Delta p)^2\rangle_\alpha = \langle p^2\rangle_\alpha - \langle p\rangle_\alpha^2 &= \frac{\hbar m\omega}{2}
\end{aligned}$$

and accordingly

$$\langle(\Delta x)^2\rangle_\alpha \langle(\Delta p)^2\rangle_\alpha = \frac{\hbar}{4} \tag{6}$$

So the states $|\alpha\rangle$, defined by (6), satisfies the minimum uncertainty relation. They are called *coherent states* and we shall now proceed to study them in detail.

2. Coherent states in the n-representation

In the $|n\rangle$ base the coherent state look like:

$$|\alpha\rangle = \sum_n c_n|n\rangle = \sum_n |n\rangle\langle n|\alpha\rangle \tag{7}$$

Since

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{8}$$

we have

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle \tag{9}$$

and thus

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{10}$$

The constant $\langle 0|\alpha\rangle$ is determined by normalization as follows:

$$1 = \sum_n \langle\alpha|n\langle\rangle n|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}$$

solving for $\langle 0|\alpha\rangle$ we get:

$$\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \tag{11}$$

up to a phase factor. Substituting into (10) we obtain the final form:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{12}$$

Obviously $|\alpha\rangle$ are *not* stationary states of the harmonic oscillator, but we shall see that they are the appropriate states for taking the classical limit.

A very convenient expression can be derived by using the explicit expression (8) for $|n\rangle$:

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(a^\dagger)^n|0\rangle = e^{\alpha a^\dagger}|0\rangle$$

which implies

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2 + \alpha a^\dagger}|0\rangle = e^{\alpha a^\dagger - \alpha^\star a}|0\rangle \tag{13}$$

3. Orthogonality and completeness relations

We proceed to calculate the overlap between the coherent states using (12).

$$\begin{aligned}\langle\alpha|\beta\rangle &= \sum_n \langle\alpha|n\rangle|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_n \frac{(\alpha^\star\beta)^n}{n!} \\ &= \exp\left(-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha^\star\beta\right) \end{aligned} \tag{14}$$

and similarly

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\beta^\star\alpha\right) \tag{15}$$

so

$$|\langle\alpha|\beta\rangle|^2 = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = \exp\left(-|\alpha|^2-|\beta|^2+\alpha^\star\beta+\alpha\beta^\star\right)$$

or

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \quad . \tag{16}$$

Since $\langle\alpha|\beta\rangle \neq 0$ for $\alpha \neq \beta$, we say that the set $\{|\alpha\rangle\}$ is *overcomplete*. There is still, however, a closure relation:

$$\int d^2\alpha\,|\alpha\rangle\langle\alpha| = \int d^2\alpha\, e^{-|\alpha|^2}\sum_{m,n}\frac{(\alpha^\star)^n\alpha^m}{\sqrt{n!\,m!}}|m\rangle\langle n| \tag{17}$$

where the measure $d^2\alpha$ means "summing" over all complex values of α, *i.e.* integrating over the whole complex plane. Now, writing α in polar form:

$$\alpha = re^{i\phi} \quad\Rightarrow\quad d^2\alpha = d\phi\,dr\,r \tag{18}$$

we get

$$\begin{aligned}\int d^2\alpha\, e^{-|\alpha|^2}(\alpha^\star)^n\alpha^m &= \int_0^\infty dr\, re^{-r^2}r^{m+n}\int_0^{2\pi}d\phi\, e^{i(m-n)\phi} \\ &= 2\pi\delta_{m,n}\frac{1}{2}\int_0^\infty dr^2\,(r^2)^m e^{-r^2} = \pi m!\delta_{m,n}\end{aligned}$$

Using this we finally get:

$$\int d^2\alpha\,|\alpha\rangle\langle\alpha| = \pi\sum_n |n\rangle\langle n| = \pi$$

or equivalently

$$\int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = 1 \tag{19}$$

4. Coherent states in the x-representation

Remember that $\langle x'|0\rangle$ is a minimal gaussian wave packet with $\langle x\rangle = \langle p\rangle = 0$. Since $\langle x'|\alpha\rangle$ is also a minimal wave packet and

$$\Re(\alpha) = \langle\alpha|\frac{a+a^\dagger}{2}|\alpha\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle\alpha|x|\alpha\rangle \tag{20}$$

$$\Im(\alpha) = \langle\alpha|\frac{a-a^\dagger}{2i}|\alpha\rangle = \frac{1}{\sqrt{2m\omega\hbar}}\langle\alpha|p|\alpha\rangle \tag{21}$$

or

$$\begin{cases} \langle x\rangle_\alpha = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\Re(\alpha) \\ \langle p\rangle_\alpha = \sqrt{m\omega\hbar}\sqrt{2}\Im(\alpha) \end{cases} \tag{22}$$

it is natural to expect that $\langle x'|\alpha\rangle$ is a displaced gaussian moving with velocity $v = p/m$. We now proceed to show this. Using the previously derived form (13) of the coherent states expressed in terms of the ground state of the harmonic oscillator we get

$$\langle x'|\alpha\rangle = \langle x'|e^{-\frac{1}{2}|\alpha|^2+\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle x'|e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}(x-\frac{ip}{m\omega})}|0\rangle$$

Acting from the left with $\langle x'|$ gives us [1]:

$$\begin{aligned} \langle x'|\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}(x'-\frac{i}{m\omega}(-i\hbar\frac{d}{dx'}))}\langle x'|0\rangle \\ &= Ne^{-\frac{1}{2}|\alpha|^2}e^{\frac{\alpha}{x_0\sqrt{2}}(x'-x_0^2\frac{d}{dx'})}e^{-\frac{1}{2}(\frac{x'}{x_0})^2} \end{aligned} \tag{23}$$

where we have used the explicit form of $\langle x'|0\rangle$ and introduced the constants:

$$\begin{cases} x_0 = \sqrt{\frac{\hbar}{m\omega}} \\ N = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \end{cases} \tag{24}$$

For notational simplicity we also put $y' = x'/x_0$. With these substitutions equation (23) will look like:

$$\langle x'|\alpha\rangle = Ne^{-\frac{1}{2}|\alpha|^2}e^{\frac{\alpha}{\sqrt{2}}(y'-\frac{d}{dy'})}e^{-\frac{1}{2}y'^2} \tag{25}$$

Using the commutator relation

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^Ae^B \tag{26}$$

[1] c.f. Sakurai p.93

which is valid if both A and B commute with $[A, B]$, we get

$$e^{\frac{\alpha}{\sqrt{2}}(y' - \frac{d}{dy'})} e^{-\frac{1}{2}y'^2} = e^{-\frac{|\alpha|^2}{4} + \frac{\alpha}{\sqrt{2}}y'} e^{-\frac{\alpha}{\sqrt{2}}\frac{d}{dy'}} e^{-\frac{1}{2}y'^2} \tag{27}$$

and thus, noting that $e^{a\frac{d}{dy'}}$, is a translation operator

$$\begin{aligned} \langle x'|\alpha\rangle &= Ne^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}y'^2 - \frac{1}{2}\alpha^2 + \sqrt{2}\alpha y'} \\ &= N\exp\left(-\frac{1}{2}(y' - \sqrt{2}\Re(\alpha))^2 + i\sqrt{2}\Im(\alpha)y' - i\Im(\alpha)\Re(\alpha)\right) \end{aligned} \tag{28}$$

Using (22) the resulting expression for the wavefunction of the coherent state is

$$\langle x'|\alpha\rangle = Ne^{-\frac{m\omega}{2\hbar}(x' - \langle x\rangle_\alpha)^2 + \frac{i}{\hbar}\langle p\rangle_\alpha x' - \frac{i}{2\hbar}\langle p\rangle_\alpha\langle x\rangle_\alpha} \tag{29}$$

and since the last term is a constant phase it can be ignored and we finally get

$$\psi_\alpha(x') = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{i}{\hbar}\langle p\rangle_\alpha x' - \frac{m\omega}{2\hbar}(x' - \langle x\rangle_\alpha)^2} \quad , \tag{30}$$

which is the promissed result.

5. Time evolution of coherent states

The time evolution of a state is given by the *time evolution operator* [2] $\mathcal{U}(t)$. Using what we know about this operator and what we have learned so far about the coherent states we can write:

$$|\alpha, t\rangle = \mathcal{U}(t,0)|\alpha(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha(0)\rangle = e^{-\frac{i}{\hbar}Ht}e^{-\frac{1}{2}|\alpha(0)|^2}\sum_n \frac{(\alpha(0))^n}{\sqrt{n!}}|n\rangle \tag{31}$$

But the $|n\rangle$:s are eigenstates of the hamiltonian so:

$$|\alpha, t\rangle = e^{-\frac{1}{2}|\alpha(0)|^2}\sum_n \frac{(\alpha(0))^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\omega\hbar(n+\frac{1}{2})t}\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{32}$$

which is the same as

$$\begin{aligned} |\alpha, t\rangle = e^{-\frac{1}{2}|\alpha(0)|^2}e^{-\frac{i}{2}\omega t}\sum_n \frac{(\alpha(0)e^{-i\omega t}a^\dagger)^n}{n!}|0\rangle = \\ \exp\left(-\frac{1}{2}|\alpha(0)|^2 - \frac{i}{2}\omega t + \alpha(0)e^{-i\omega t}a^\dagger\right)|0\rangle \end{aligned} \tag{33}$$

[2] c.f. Sakurai chapter 2.1

Comparing this expression with (13), it is obvious that the first and the third term in the exponent, operating on the ground state, will give us a coherent state with the time dependent eigenvalue $e^{-i\omega t}\alpha(0)$ while the second term only will contribute with a phase factor. Thus we have:

$$|\alpha, t\rangle = e^{-\frac{i}{2}\omega t}|e^{-i\omega t}\alpha(0)\rangle = |\alpha(t)\rangle \tag{34}$$

So *the coherent state remains coherent under time evolution.* Furthermore,

$$\alpha(t) = e^{-i\omega t}\alpha(0) \quad \Rightarrow \quad \frac{d}{dt}\alpha(t) = -i\omega\alpha(t) \tag{35}$$

or in components

$$\begin{cases} \frac{d}{dt}\Re(\alpha) = \omega\Im(\alpha) \\ \frac{d}{dt}\Im(\alpha) = -\omega\Re(\alpha) \end{cases} \tag{36}$$

Defining the expectation values,

$$\begin{cases} x(t) = \langle\alpha(t)|x|\alpha, t\rangle \\ p(t) = \langle\alpha(t)|p|\alpha, t\rangle \end{cases} \tag{37}$$

we get

$$\begin{cases} \frac{d}{dt}x(t) = \sqrt{\frac{\hbar}{2m\omega}}2\frac{d}{dt}\Re(\alpha) = \sqrt{\frac{\hbar}{2m\omega}}2\omega\Im(\alpha) = \frac{p(t)}{m} \\ \frac{d}{dt}p(t) = i\sqrt{\frac{m\hbar\omega}{2}}(-2i)\frac{d}{dt}\Im(\alpha) = -\sqrt{\frac{m\hbar\omega}{2}}2\omega\Re(\alpha) = -m\omega^2 x(t) \end{cases} \tag{38}$$

or in a more familiar form

$$\begin{cases} p(t) = m\frac{d}{dt}x(t) = mv(t) \\ \frac{d}{dt}p(t) = -m\omega^2 x(t) \end{cases} \tag{39}$$

i.e. x(t) and p(t) satisfies the classical equations of motion, as expected from Ehrenfest's theorem.

In summary, we have seen that the coherent states are minimal uncertainty wavepackets which remains minimal under time evolution. Furthermore, the time dependant expectation values of x and p satifies the classical equations of motion. From this point of view, the coherent states are very natural for studying the classical limit of quantum mechanics. This will be explored in the next part.